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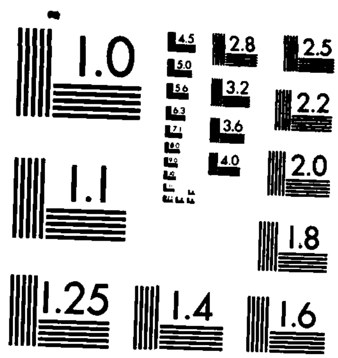

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1. INTRODUCTION

The problem of testing for the dimensionality of the regression matrix under multivariate regression model has received considerable attention in the literature. This problem arises in the areas of pattern recognition, signal processing, and functional and structural relations. For a discussion on applications in functional relations, the reader is referred to Anderson(1984).

Fisher (1938) considered the problem of testing for the number of significant discriminant functions and it is a special case of the problem of testing for the rank of the regression matrix. Tintner (1945) derived the likelihood ratio test (LRT) statistic for the rank of the above matrix when the covariance matrix is known. Anderson (1951) derived the(LRT) statistic for the rank of the regression matrix when the covariance matrix is known. Fujikoshi (1974) derived the LRT procedure for the rank of the regression matrix under growth curve model. Recently, Rao (1985) considered the LRT procedure for the rank under a general model, incorporating the multivariate regression model and the two-way classification with interaction and with one observation per cell. The above work was done when the underlying distribution is multivariate normal. The object of this paper is to discuss various procedures for testing for the dimensionality of the regression matrices and derive asymptotic distributions of the test statistics when the underlying distribution is real or complex elliptically symmetric distribution.

In Section 2 of this paper, we give some preliminaries and state the main problems that are considered. The LRT procedures for the dimensionality of the regression matrices are derived in Sections 3 and 4 for the cases of the real and complex elliptically symmetric distributions respectively. Asymptotic distributions of the above test statistics are derived in Section 5 when the joint distribution of the observations is elliptically symmetric. Multivariate normal and multivariate t distributions are special cases of the elliptically symmetric distributions. In Section 6, we derive the asymptotic distribution of the LRT statistic for the rank of the regression matrix when the observations are distributed indepently as elliptically symmetric. The assumptions made about the underlying distributions is Sections 5 and 6 are equivalent only in the case of multivariate normal.

2. STATEMENT OF PROBLEMS AND PRELIMINARIES

Consider the model

$$X = A\theta + E \tag{2.1}$$

where the error matrix E: $n \times p$ is distributed as an elliptically symmetric distribution with density

$$f(E) = \frac{1}{|\Sigma|^{n/2}} h(\mathrm{tr}\Sigma^{-1}E'E) \tag{2.2}$$

where h(x) is strictly decreasing and differentiable function of x. Also, A: $n \times k$ is known and of rank $k < n$, and θ: $k \times p$ is unknown. Now, let

$$\Delta = C\theta \tag{2.3}$$

where C: $u \times k$ is known and of rank u. Let H_1 denote the hypothesis that the rank of Δ is r and H_2 denote the hypothesis that $\Delta \in P_r$. Here $\Delta \in P_r$ denotes that the rows of Δ lie in a r-dimensional plane in p-dimensional space.

Now, let $\pi_r(\underset{\sim}{a})$ denote the set of $n \times p$ matrices of the form

$$M = (GF + \underset{\sim}{a}\underset{\sim}{b}')D \tag{2.4}$$

where $|G'G| \neq 0$, $FF' = I_r$ and D: $p \times p$ is any positive definite matrix and $\underset{\sim}{b}$ is any $p \times 1$ vector. Then H_1 denotes the hypothesis that $\Delta \in \pi_r(\underset{\sim}{0})$ whereas H_2 denotes the hypothesis that $\Delta \in \pi_r(\underset{\sim}{1})$ where $\underset{\sim}{1}' = (1,\ldots,1)$.

Next, consider the model

$$Z = A\beta + N \tag{2.5}$$

where β: $k \times p$ is an unknown complex matrix, N: $n \times p$ is distributed as the complex elliptically symmetric distribution with density given by

$$f(N) = \frac{2^{np}}{|\Sigma|^n} h(2\,\mathrm{tr}\,\Sigma^{-1}N'\bar{N}). \tag{2.6}$$

Here, we note that the complex elliptical distribution was introduced by Krishnaiah and Lin (1984). In this paper, we consider the problems of testing the hypotheses H_1 and H_2 when the underlying distributions are real and complex elliptically symmetric. We need the following lemmas in the sequel.

Lemma 2.1. If $\underset{\sim}{a}$ is a u×1 complex vector, L: n×p is a complex matirx, Q: p×p is Hermitian, positive definite matrix such that the rank of

$$\hat{S} = Q^{-1/2}L'(I_u-\bar{\underset{\sim}{a}}(\underset{\sim}{a}'\bar{\underset{\sim}{a}})^{-1}\underset{\sim}{a}')\bar{L}Q^{-1/2}$$

is r or more, then the eigenvalues of

$$S(M) = Q^{-1/2}(L-M)'(\overline{L-M})Q^{-1/2}$$

are minimized simultaneously with respect to $M \in \pi_r(\underset{\sim}{a})$ if and only if

$$M = [\underset{\sim}{a}(\bar{\underset{\sim}{a}}'\underset{\sim}{a})^{-1}\bar{\underset{\sim}{a}}'L\bar{Q}^{-1/2}+(I_n-\underset{\sim}{a}(\bar{\underset{\sim}{a}}'\underset{\sim}{a})'\bar{\underset{\sim}{a}}')L\bar{Q}^{-1/2}\bar{V}_r'V_r]\bar{Q}^{1/2}$$

where the rows of V_r consist of normalized eigenvectors corresponding to the first r largest eigenvalues of $\hat{S}$. The minimum values of $ch_i(S(M))$ are given by ϕ_{r+i} where $ch_i(A)$ denotes i-th largest eigenvalue of A, $\phi_1 \geq \ldots \geq \phi_p$ are the eigenvalues of $\hat{S}$ and $\phi_j = 0$ for $j \geq p$.

When $\underset{\sim}{a}$, L and Q are real, the above lemma was proved by Fujikoshi (1974). The proof of Lemma 2.1 follows along the same lines as in Fujikoshi (1974). From Lemma 2.1, the following lemma follows immediately:

Lemma 2.2. Let f(S) be a function of a p×p Hermitian matrix S such that

$$f(S) = g(ch_1(S),\ldots,ch_p(S))$$

and $g(\cdot)$ is strictly decreasing in each argument. Then

$$\max_{M \in \pi_r(\underset{\sim}{a})} f(S(M)) = g(\phi_{r+1}, \ldots, \phi_p, 0, \ldots, 0)$$

where $\phi_j = ch_j(S)$ for $j = 1,2,\ldots,p$.

The following lemma was proved in Bai (1984).

Lemma 2.3. Suppose

$$f_n(Z) = a_K^{(n)} Z^K + a_{K-1}^{(n)} Z^{K-1} + \ldots + a_0^{(n)}$$

$$f(Z) = a_k Z^k + a_{k-1} Z^{k-1} + \ldots + a_0$$

where $K \geq k$. Also, let $f_n(Z) \to f(Z)$ as $n \to \infty$, where $a_K^{(n)} \neq 0$, $n = 1,2,\ldots$ and $a_k \neq 0$. In addition, let $Z_1,\ldots,Z_k$ denote the roots of $f(Z)$. Then, we can suitably arrange the roots of f_n as $Z_1^{(n)},\ldots,Z_k^{(n)},\ldots,Z_K^{(n)}$ such that

$$Z_i^{(n)} \to Z_i \quad \text{for } i \leq k$$

$$|Z_i|^n \to \infty \quad \text{for } i > k$$

as $n \to \infty$.

3. LR TESTS FOR THE DIMENSIONALITY OF THE REGRESSION MATRIX

In this section, we derive the LR test for the dimensionality of the regression matrix under the model (2.1) for the cases when Σ is known and unknown and the underlying distribution is elliptically symmetric. For the sake of simplicity, we first reduce the model to a canonical form. It is known that nonsingular matrices $\underset{k\times k}{T_A}$, $\underset{u\times u}{T_C}$ and orthogonal matrices $\underset{n\times n}{\Gamma_A}$, $\underset{k\times k}{\Gamma_C}$ exists such that

$$\underset{n\times k}{A} = \Gamma_A \begin{pmatrix} I_k \\ 0 \end{pmatrix} T_A \tag{3.1}$$

$$CT_A^{-1} = T_C(I_u \;\; 0)\Gamma_C . \tag{3.2}$$

We make the following orthogonal transformation

$$Y - \Xi = \begin{pmatrix} \Gamma_C & 0 \\ 0 & I_{n-k} \end{pmatrix} \Gamma_A' \; (X - A\theta) . \tag{3.3}$$

From assumption (2.2) we get the density function of Y as

$$|\Sigma|^{-\frac{n}{2}} h(\mathrm{tr}\Sigma^{-1}(Y-\Xi)'(Y-\Xi)). \tag{3.4}$$

If we partition Y and Ξ as

$$Y = \underset{p}{\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}} \begin{matrix} u \\ k-u \\ n-k \end{matrix} \qquad \Xi = \underset{p}{\begin{pmatrix} \Xi_1 \\ \Xi_2 \\ \Xi_3 \end{pmatrix}} \begin{matrix} u \\ k-u \\ n-k \end{matrix}$$

it is easy to see that

$$Y_1 = (I_u\ 0)\Gamma_C(I_k\ 0)\Gamma_A'X$$

$$Y_2 = (0\ I_{k-u})\Gamma_C(I_k\ 0)\Gamma_A'X$$

$$Y_3 = (0\ I_{n-k})\Gamma_A'X$$

$$\Xi_1 = (I_n\ 0)\Gamma_C(I_k\ 0)\Gamma_a'A\theta = T_C^{-1}\Delta \tag{3.5}$$

$$\Xi_2 = (0\ I_{k-u})\Gamma_C(I_k\ 0)\Gamma_A'A\theta = (0\ I_{k-u})T_C T_A\theta$$

$$\Xi_3 = (0\ I_{n-k})\Gamma_A'A\theta = (0\ I_{n-k})\begin{pmatrix} I_k \\ 0 \end{pmatrix} T_A\theta = 0.$$

Under canonical form the hypotheses H_1 and H_2 are equivalent to $\Xi_1 \in \pi_r(\underset{\sim}{0})$ and $\Xi_1 \in \pi_r(\underset{\sim}{\alpha})$ respectively where $\underset{\sim}{\alpha} = T_C^{-1}\underset{\sim}{1}$. Now, let H_1^* and H_2^* denote the alternative hypotheses $\Xi_1 \in \pi_{r'}(\underset{\sim}{0})$ and $\Xi_1 \in \pi_{r'}(\underset{\sim}{\alpha})$ respectively for some $r' > r$. Also, let

$$\begin{aligned}
M &= C(A'A)^{-1}C' \\
\Xi_0 &= (A'A)^{-1}A'X \\
S_h(\Xi,M) &= (C\,\Xi)'M^{-1}(C\Xi) \\
S_f(\Xi,M) &= (C\Xi)'\{M^{-1}-M^{-1}\underset{\sim}{1}(\underset{\sim}{1}'M^{-1}\underset{\sim}{1})^{-1}\underset{\sim}{1}'M^{-1}\}C\Xi \\
S &= X'(I-A(A'A)^{-1}A')X.
\end{aligned} \tag{3.6}$$

3.1 LRT Statistics when Σ is Known

When Σ is known, the LRT statistic for testing H_1 against H_1^* is given by

$$T_1 = \frac{\max\limits_{\Xi_1 \in \pi_r(\underset{\sim}{0})} \sup\limits_{\Xi_2} |\Sigma|^{-\frac{n}{2}} h(\mathrm{tr}\Sigma^{-1}(Y-\Xi)'(Y-\Xi))}{\sup\limits_{\Xi_1,\Xi_2} |\Sigma|^{-\frac{n}{2}} h(\mathrm{tr}\Sigma^{-1}(Y-\Xi)'(Y-\Xi))}$$

$$= \frac{\max_{\Xi_1 \in \pi_r(\underset{\sim}{0})} h(\mathrm{tr}\Sigma^{-1}(Y_1-\Xi_1)'(Y_1-\Xi_1)+\mathrm{tr}\Sigma^{-1}Y_3'Y_3)}{h(\mathrm{tr}\Sigma^{-1}Y_3'Y_3)} .$$

Since h(x) is a decreasing function of x, $Y_3\Sigma^{-1}Y_3' \geq 0$ and $(Y_1-\Xi_1)\Sigma^{-1}(Y_1-\Xi)' \geq 0$, we obtain, from Lemma 2.1,

$$T_1 = \frac{h(\phi_{r+1}+\ldots+\phi_s+\mathrm{tr}\Sigma^{-1}Y_3'Y_3)}{h(\mathrm{tr}\Sigma^{-1}Y_3'Y_3)} \tag{3.7}$$

where $Y_3'Y_3 = X'(I_n - A(A'A)^{-1}A')X$, $\phi_1 \geq \ldots \geq \phi_s > 0$ are positive eigenvalues of $\Sigma^{-1/2}Y_1'Y_1\Sigma^{-1/2} = S_h(\Xi_0,M)\Sigma^{-1}$ and $s = \min(u,p)$. When the underlying distribution is multivariate normal, we obtain

$$-2\log T_1 = \phi_{r+1}+\ldots+\phi_s . \tag{3.8}$$

The LRT statistic for testing H_2 against H_2^* is given by

$$T_2 = \frac{\max_{\Xi_1 \in \pi_r(\underset{\sim}{\alpha})} h(\mathrm{tr}\Sigma^{-1}(Y_1-\Xi_1)'(Y_1-\Xi_1)+\mathrm{tr}\Sigma^{-1}Y_3'Y_3)}{h(\mathrm{tr}\Sigma^{-1}Y_3'Y_3)}$$

$$= \frac{h(\psi_{r+1}+\ldots+\psi_{\bar{s}} + \mathrm{tr}\Sigma^{-1}Y_3'Y_3)}{h(\mathrm{tr}\Sigma^{-1}Y_3'Y_3)} \tag{3.9}$$

where $\psi_1 \geq \ldots \geq \psi_{\bar{s}} > 0$ are the positive eigenvalues of $S_f(\Xi_0,M)\Sigma^{-1}$ and $\bar{s} = \min(u-1,p)$. When the underlying distribution is multivariate normal, we obtain

$$-2\log T_2 = \psi_{r+1}+\ldots+\psi_{\bar{s}} . \tag{3.10}$$

3.2 LRT Statistics when Σ is Unknown

When Σ is unknown, the LRT statistic for testing H_1 against H_1^* is given by

$$T_3 = \frac{\max\limits_{\Xi_1 \epsilon \pi_r(\underset{\sim}{0})} \max\limits_{\Sigma>0} \sup\limits_{\Xi_2} |\Sigma|^{-\frac{n}{2}} h(\mathrm{tr}\Sigma^{-1}(Y-\Xi)'(Y-\Xi))}{\max\limits_{\Sigma>0} \sup\limits_{\Xi_2} |\Sigma|^{-\frac{n}{2}} h(\mathrm{tr}\Sigma^{-1}(Y-\Xi)'(Y-\Xi))}$$

$$= \frac{\max\limits_{\Xi_1 \epsilon \pi_r(\underset{\sim}{0})} \max\limits_{\Sigma>0} |\Sigma|^{-\frac{n}{2}} h(\mathrm{tr}\Sigma^{-1}(Y_1-\Xi_1)'(Y_1-\Xi)+Y_3'Y_3))}{\max\limits_{\Sigma>0} |\Sigma|^{-\frac{n}{2}} h(\mathrm{tr}\Sigma^{-1}Y_3'Y_3)}$$

$$= \frac{\max\limits_{\Xi_1 \epsilon \pi_r(\underset{\sim}{0})} \lambda_{\max(h)}^{-\frac{np}{2}} \left|(Y_1-\Xi_1)'(Y_1-\Xi_1)+Y_3'Y_3\right|^{-\frac{n}{2}} h(\frac{p}{\lambda_{\max(h)}})}{\lambda_{\max(h)}^{-\frac{np}{2}} |Y_3'Y_3|^{-\frac{n}{2}} h(\frac{p}{\lambda_{\max(h)}})} . \tag{3.11}$$

Equation (3.11) follows from Anderson and Fang (1982) who derived the LRT procedure for $\theta = 0$ when the underlying distribution is real elliptically symmetric. Now, using Lemma 2.1, we obtain

$$T_3 = \max_{\Xi_1 \epsilon \pi_r(\underset{\sim}{0})} \left| I-(Y_3'Y_3)^{-\frac{1}{2}}((Y_1-\Xi_1)'(Y_1-\Xi_1))(Y_3'Y_3)^{-\frac{1}{2}} \right|^{-\frac{n}{2}}$$

$$= ((1+d_{r+1})(1+d_{r+2})\dots(1+d_s))^{-\frac{n}{2}} \tag{3.12}$$

where $s = \min(u,p)$ and $d_1 \geq \dots \geq d_s > 0$ are the positive eigenvalues of $S_h(\Xi_0,M)S^{-1}$.

Similarly, the LRT statistic for testing H_2 against H_2^* is

$$T_4 = \max_{\Xi_1 \epsilon \pi_r(\underset{\sim}{1})} \left| I-(Y_3'Y_3)^{-\frac{1}{2}}((Y_1-\Xi_1)'(Y_1-\Xi_1))(Y_3'Y_3)^{-\frac{1}{2}} \right|^{-\frac{n}{2}} \tag{3 13}$$

$$= [(1+\ell_{r+1})\dots(1+\ell_{\bar{s}})]^{-\frac{n}{2}} \tag{3.14}$$

where $\bar{s} = \min(u-1,p)$ and $\ell_1 \geq \dots \geq \ell_{\bar{s}} > 0$ are the positive eigenvalues of $S_f(\Xi_0,M)S^{-1}$.

When the underlying distribution is multivariate normal, the test statistics T_3 and T_4 were derived by Fujikoshi (1974) and T_2 was derived by Rao (1965).

4. LRT STATISTICS FOR THE DIMENSIONALITY OF REGRESSION MATRIX IN COMPLEX ELLIPTICAL CASE

Consider the complex multivariate regression model (2.5) where N is distributed as (2.6). Also, let

$$\Delta_0 = C\Theta \tag{4.1}$$

where C: u×k is known and is of rank u. In addition, let H_{10} denote the hypotheses that the rank of Δ_0 is r whereas H_{10}^* denote the alternative hypothesis that the rank of Δ_0 is greater than r where $r < s = \min(u,p)$. Also, let H_{20} denote the hypothesis that $\Delta_0 \in P_r$ and let H_{20}^* denote the alternative hypothesis that $\Delta_0 \in P_{r'}$ for some $r' > r$ and $r < \bar{s} = \min(u-1,p)$. Here $\Delta \epsilon P_r$ means that the rows of Δ_0 actually lie in a r dimensional complex plane. The hypotheses H_{10} and H_{20} are respectively equivalent to $\Delta \in \pi_r(\underset{\sim}{0})$ and $\Delta \in \pi_r(\underset{\sim}{1})$. Also, let H_{10}^* denote the alternative hypothesis that $\Delta \in \pi_r(\underset{\sim}{0})$ for some $r' > r$ and H_{20}^* denote the alternative hypothesis that $\Delta \in \pi_r(\underset{\sim}{1})$ for some $r' > r$. We now reduce the model in canonical form as in the real case.

The problem of testing H_{10} against H_{10}^* is equivalent to testing the hypothesis $\Xi_1 \in \pi_r(\underset{\sim}{0})$ against the alternative $\Xi_1 \in \pi_{r'}(\underset{\sim}{0})$ for some $r' > r$ and $r < s = \min(u,p)$ in the canonical form. Similarly, the problem of testing H_{20} against H_{20}^* is equivalent to testing the hypothesis $\Xi_1 \in \pi_r(\underset{\sim}{\alpha})$ against the alternative $\Xi_1 \in \pi_{r'}(\underset{\sim}{\alpha})$ for some $r' > r$ and $\underset{\sim}{\alpha} = T_C^{-1}\underset{\sim}{1}$ and $r < \bar{s} = \min(u-1,p)$. When Σ is known, let T_5 denote the LRT statistic for testing H_{10} against H_{10}^* and let T_6 denote the LRT statistic for testing H_{20} against H_{20}^*. Then, using Lemma 2.1, we obtain the following:

$$T_5 = \frac{h(2\phi_{r+1}+\dots+2\phi_s + 2\,\mathrm{tr}\,\Sigma^{-1}Y_3'\bar{Y}_3}{h(2\,\mathrm{tr}\,\Sigma^{-1}Y_3'\bar{Y}_3)} \tag{4.2}$$

$$T_6 = \frac{h(2\psi_{r+1}+\ldots+2\psi_s + 2\,\mathrm{tr}\,\Sigma^{-1}Y_3'\bar{Y}_3)}{h(2\,\mathrm{tr}\,\Sigma^{-1}Y_3'\bar{Y}_3)} \tag{4.3}$$

where $\phi_1 \geq \ldots \geq \phi_s$ are the nonzero eigenvalues of $S_h(\hat{\theta},M)S_\ell(I_p,M)^{-1}$, and $\psi_1 \geq \ldots \geq \psi_s$ are the nonzero eigenvalues of $S_f(\hat{\theta},M)S_\ell(I_p,M)^{-1}$. Here

$$\begin{aligned} S_h(\hat{\theta},M) &= (C\hat{\theta})'M^{-1}\overline{(C\hat{\theta})} \\ S_\ell(I_p,\Sigma) &= \Sigma \\ M &= \bar{C}(A'A)^{-1}C' \\ S_f(\hat{\theta},M) &= (C\hat{\theta})'[M^{-1}-M^{-1}\underset{\sim}{1}(\underset{\sim}{1}'M^{-1}\underset{\sim}{1})^{-1}\underset{\sim}{1}'M^{-1}]\overline{(C\hat{\theta})}. \end{aligned} \tag{4.4}$$

When the underlying distribution is complex multivariate normal, we have

$$T_5 = \exp\{-(\phi_{r+1}+\ldots+\phi_s)\} \tag{4.5}$$

$$T_6 = \exp\{-(\psi_{r+1}+\ldots+\psi_s)\}. \tag{4.6}$$

When Σ is unknown, we denote the LRT statistic for H_{10} and H_{20} against H_{10}^* and H_{20}^* by T_7 and T_8 respectively. Then

$$T_7 = \{(1+d_{r+1})(1+d_{r+2})\ldots(1+d_s)\}^{-n} \tag{4.7}$$

$$T_8 = \{(1+\ell_{r+1})(1+\ell_{r+2})\ldots(1+\ell_s)\}^{-n} \tag{4.8}$$

where $d_1 \geq \ldots \geq d_s$ are the nonzero eigenvalues of $S_h(\hat{\theta},M)\{S_\ell(I,S)\}^{-1}$ and $\ell_1 \geq \ldots \geq \ell_s$ are the nonzero eigenvalues of $S_f(\hat{\theta},M)\{S_\ell(I,S)\}^{-1}$.

5. ASYMPTOTIC DISTRIBUTIONS OF LRT TEST STATISTICS FOR THE DIMENSIONALITY OF REGRESSION MATRIX

We know that

$$Y \sim |\Sigma|^{-n/2} h(\mathrm{tr}\Sigma^{-1}(Y_1-\Xi_1)'(Y_1-\Xi)+(Y_2-\Xi_2)'(Y_2-\Xi)+Y_3'Y_3) \tag{5.1}$$

and $d_1 \geq \ldots \geq d_s > 0$ are the positive roots of

$$0 = |S_h(\Xi_0,M) - dS| = |Y_1'Y_1 - dY_3'Y_3|. \tag{5.2}$$

Since $\Xi_1 \in \pi_r(\underset{\sim}{0})$, i.e., $\mathrm{rk}(\Xi_1) = r > 0$, we know that

$$|\Xi_1'\Xi_1 - n\lambda\Sigma| = 0 \tag{5.3}$$

will have zero roots with multiplicity p-r and r nonzero roots. We arrange these nonzero roots in order of decreasing magnitude and they appear as

$$\lambda_{u_{n-1}+1} = \ldots = \lambda_{u_h} = \lambda_h^* \qquad h = 1,2,\ldots,\ell$$

where $u_0 = 0$, $u_1 + \ldots + u_\ell = r$, $\lambda_1 > \ldots > \lambda_\ell^*$.

Since $\Sigma^{-1} > 0$ we can write $\Sigma^{-1} = Q'Q$, $Q > 0$ and $\mathrm{rk}(Q\Xi_1) = \mathrm{rk}(\Xi_1) = r$. Hence there exist orthogonal matrices V_1, V_2, such that

$$V_1'Q\ {}_1'V_2 = \begin{bmatrix} \sqrt{n\lambda_1} I_{u_1} & 0 & \cdots & 0 \\ 0 & \sqrt{n\lambda_1} I_{u_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots 0 & \cdots & \sqrt{n\lambda_\ell} I_{u_\ell} \\ 0 & 0 & 0 & 0 \end{bmatrix} = \Lambda\ \text{(say)}$$

Let

$$QY_1' = V_1'Y_{1*}V_2 \quad QY_{2*}' = V_1'W_2V_2 \quad QY_3' = V_1'W_3V_2.$$

Then $d_1 \geq \ldots \geq d_s > 0$ are also the positive roots of

$$|Y_{1*}'Y_{1*} - dW_3'W_3| = 0$$

and the density of $\begin{pmatrix} Y_{1*} \\ W_3 \end{pmatrix}$ exists, and it is the marginal density of

$$h(\operatorname{tr}(Y_{1*}-\Lambda)'(Y_{1*}-\Lambda) + W_2'W_2 + W_3'W_3)). \tag{5.4}$$

Let $Y_{1*} = W_1+\Lambda$, then $d_1 \geq \ldots \geq d_s > 0$ satisfies

$$|(W_1+\Lambda)'(W_1+\Lambda) - dW_3'W_3| = 0 \tag{5.5}$$

and the density of $\begin{pmatrix} W_1 \\ W_3 \end{pmatrix}$ is the marginal density of

$$W = (w_{\alpha\beta}) = \begin{array}{c} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \\ p \end{array} \begin{array}{l} u \\ k-u \\ n-k \\ \end{array} \sim h(\operatorname{tr} w'w)$$

$$= h(\operatorname{tr}(w_1'w_1 + w_2'w_2 + w_3'w_3)). \tag{5.6}$$

Let

$$\underset{u\times p}{W_1} = (w_{1\alpha\beta}) \qquad \underset{(n-k)\times p}{W_3} = (w_{3\alpha\beta})$$

$$A = W_1'W_1 \qquad B = W_3'W_3$$

$$C = W_1' \left[\begin{array}{cccc} \sqrt{\lambda_1}\, I_{u_1} & 0 & \ldots & 0 \\ 0 & \sqrt{\lambda_2} I_{u_2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots\ 0 & \ldots & \sqrt{\lambda_\ell} I_{u_\ell} \\ 0 & \ldots\ 0 & \ldots & 0 \end{array} \right] + \left[\begin{array}{cccc} \sqrt{\lambda_1} I_{u_1} & 0 & \ldots & 0 \\ 0 \ldots & \sqrt{\lambda_2} I_{u_2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 \ldots & 0 & \ldots & \sqrt{\lambda_\ell} I_{u_\ell} \\ 0 \ldots & 0 & \ldots & 0 \end{array} \right]' W_1$$

$$= \begin{bmatrix} C_{11} & \cdots & C_{1\ell} & E_1 \\ \vdots & & \vdots & \vdots \\ C_{\ell 1} & \cdots & C_{\ell\ell} & E_\ell \\ E_1' & \cdots & E_\ell' & 0 \end{bmatrix}$$

where

$$E_h = \sqrt{\lambda_h}\,(w_{1\alpha\beta}) \qquad \alpha = u_{h-1}+1,\ldots,u_h \quad .$$
$$\beta = r+1,\ldots,u\,.$$

Then (5.5) can be written as

$$\left|n^{-1}A + n^{-1/2}C + \begin{pmatrix} \lambda_1 I_{u_1} & \cdot & \cdot & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & \cdot & \lambda_\ell I_{u_\ell} \\ 0 & & & 0 \end{pmatrix} - d\,\frac{1}{n}B\right| = 0\,. \tag{5.7}$$

Let $d_1,\ldots,d_s$ be the roots of (5.7). We classify them into $\ell+1$ sets, containing $u_1,\ldots,u_\ell$, s-r members respectively.

For the last set let d^* be any one of them putting $d = n^{-1}\tau$ and substituting it in (3.19), cancelling the common factor $n^{-1/2}$ from the last p-r rows and columns and finally letting $n \to \infty$. It is equivalent to

$$\lim_{n\to\infty} \begin{vmatrix} \lambda_1 I_{u_1} & \cdot & \cdot & \cdot & \cdot & \cdot & E_1 \\ \cdot & \ddots & & & & & \vdots \\ \cdot & & \lambda_\ell I_{u_\ell} & & & & E_\ell \\ \cdot & & & & & & \\ E_1 & \cdots & E_\ell & & & & A_{\ell+1\ell+1} - \tau\,\frac{1}{n}\,B_{\ell+1\ell+1} \end{vmatrix} = 0 \tag{5.8}$$

where

$$A_{\ell+1\ell+1} = \Big(\sum_{t=1}^{u} w_{1t\alpha}w_{1t\beta}\Big) \qquad \alpha,\beta = r+1,\ldots,p.$$

$$B_{\ell+1\ell+1} = \Big(\sum_{t=1}^{n-k} w_{3t\alpha}w_{3t\beta}\Big) \qquad \alpha,\beta = r+1,\ldots,p.$$

Equation (5.8) is equivalent to

$$\lim_{n\to\infty} \left|A_{\ell+1\ell+1} - \frac{1}{\lambda_1}E_1'E_1 - \ldots - \frac{1}{\lambda_\ell}E_\ell'E_\ell - \tau\,\frac{1}{n}\,B_{\ell+1\ell+1}\right|$$
$$= \lim_{n\to\infty} \left|E - \tau\frac{1}{n}\,B_{\ell+1\ell+1}\right| = 0 \tag{5.9}$$

where

$$E = \left(\sum_{t=r+1}^{u} w_{1t\alpha} w_{1t\beta}\right) \qquad \alpha,\beta = r+1,\dots,p.$$

Here, we note that eq. (5.8) is obtained by following same lines as in Hsu (1941). When the underlying distribution is multivariate normal, we denote E, $B_{\ell+1\ell+1}$ by $E^{(N)}$ and $B^{(N)}_{\ell+1\ell+1}$ respectively. We have

$$n\, E^{(N)} B^{(N)^{-1}}_{\ell+1,\ell+1} \overset{d}{=} n\, E\, B^{-1}_{\ell+1,\ell+1} \,.$$

Since

$$\lim_{n\to\infty} \frac{1}{n} B^{(N)}_{\ell+1\ell+1} = I \quad \text{(a.e.)}$$

we have

$$\lim_{n\to\infty} \left| E^{(N)} - \tau \frac{1}{n} B_{\ell+1\ell+1} \right| = \left| E^{(N)} - \tau I \right| = 0 \quad \text{(a.e.)} \tag{5.10}$$

(5.10) has s-r nonzero roots, written as $\tau_{r+1},\dots,\tau_s$. Let

$$\bar{\tau}_i = n d_i \qquad i = r+1,\dots,s.$$

Then, by Lemma 2.3, for $i = r+1,\dots,s$ we have

$$\bar{\tau}_i \to \tau_i \qquad \text{as} \quad n \to \infty.$$

So when $n \to \infty$

$$\begin{aligned} -2 \ln T_3 &= -2 \sum_{i=r+1}^{s} \ln(1+d_i)^{-\frac{n}{2}} \\ &= \sum_{i=r+1}^{s} \ln\left(1 + \frac{\bar{\tau}_i}{n}\right)^n \\ &\to \sum_{i=r+1}^{s} \tau_i = \operatorname{tr} E^{(N)} \sim \chi^2_{(p-r)(u-r)}. \end{aligned} \tag{5.11}$$

Similarly, we can show that the asymptotic null distribution of $-2 \ln T_4$ is $\chi^2_{(p-r)(u-r-1)}$.

We know $T_4 = ((1+\ell_{r+1})\ldots(1+\ell_{\bar{s}}))^{-\frac{n}{2}}$, $\bar{s} = \min(u-1,p)$ and $\ell_1 \geq \ldots \geq \ell_{\bar{s}} > 0$ are the positive roots of

$$0 = |S_f(\Xi_0 M) - \ell S|$$
$$= |Y'[I_u-(T_C^{-1}1)((T_C^{-1}1)'(T_C^{-1}1))^{-1}(T_C^{-1})']Y - \ell Y_3'Y_3|$$

where

$\begin{pmatrix} Y_1 \\ Y_3 \end{pmatrix}$ has the same density as in the proof of (5.11). Since

$$\mathrm{rk}[I_u-(T_C^{-1}1)((T_C^{-1}1)'(T_C^{-1}1))^{-1}(T_C^{-1}1)'] = u-1$$

and it is an idempotent matrix, there exists an orthogonal matrix $\Gamma_{u\times u}$ such that

$$\Gamma[I_u-(T_C^{-1}1)((T_C^{-1}1)'(T_C^{-1}))^{-1}(T_C^{-1}1)') = [\begin{smallmatrix} I_{u-1} & 0 \\ 0 & 0 \end{smallmatrix}].$$

Taking orthogonal transformation

$$\begin{pmatrix} X_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} \Gamma & 0 \\ 0 & I_{n-u} \end{pmatrix} Y = \begin{pmatrix} \Gamma Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

i.e.

$$Y_1 = \Gamma' X \quad Y_2 = Y_2 \quad Y_3 = Y_3 .$$

Partition X as

$$X = \begin{array}{c} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{matrix} u-1 \\ 1 \end{matrix} \\ p \end{array} .$$

Then $\ell_1 \geq \ldots \geq \ell_{\bar{s}} > 0$ are positive roots of

$$|X_1'X_1 - \ell Y_3'Y_3| = 0$$

and the density of $\begin{pmatrix} X_1 \\ Y_3 \end{pmatrix}$ is the marginal density of

$$|\Sigma|^{-\frac{n}{2}} h[\operatorname{tr}^{-1}(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} - \Gamma'\Xi_1)'(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} - \Gamma'\Xi_1) + Y_{2*}'Y_{2*} + Y_3'Y_3]$$

where

$$Y_{2*} = Y_2 + \Xi_2.$$

Since $\operatorname{rk}(\Gamma'\Xi_1) = r > 0$. Similar to (5.11), there exist orthogonal matrices V_1, V_2 such that

$$V_1'(Q\Gamma'\Xi_1)V_2 = \begin{pmatrix} \sqrt{n\lambda_1}\, I_{u_1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \sqrt{n\lambda_\ell}\, I_{u_\ell} \\ & 0 & \end{pmatrix} = \begin{matrix} u-1 \\ \\ \end{matrix} \begin{pmatrix} \Lambda^{(1)} \\ 0 \end{pmatrix} \text{(say)}$$

where $\lambda_1 > \ldots > \lambda_\ell$ $u_1 + \ldots + u_\ell = r$ and $\Sigma^{-1} = Q'Q$ $Q > 0$. Taking transformations

$$QX' = V_1'X_*V_2 \quad QY_{2*}' = V_1'W_2V_2 \quad QY_3' = V_1'W_3V_2$$

and denoting

$$X_* = w_1 + \begin{pmatrix} \Lambda^{(1)} \\ 0 \end{pmatrix} \equiv \begin{pmatrix} w_1^{(1)} \\ w_1^{(2)} \end{pmatrix} + \begin{pmatrix} \Lambda^{(1)} \\ 0 \end{pmatrix}$$

we observe that $\ell_1 \geq \ldots \geq \ell_s > 0$ are the roots of

$$|(w_1^{(1)} + \Lambda^{(1)})'(w_1^{(1)} + \Lambda^{(1)}) - \ell w_3'w_3| = 0$$

and the density of $\begin{pmatrix} w_1^{(1)} \\ w_3 \end{pmatrix}$ is the marginal density of

$$W = (w_{\alpha\beta}) = \begin{array}{c|c} \begin{pmatrix} w_1^{(1)} \\ w_1^{(2)} \\ w_2 \\ w_3 \end{pmatrix} & \begin{matrix} u-1 \\ 1 \\ k-u \\ n-k \end{matrix} \end{array} \sim h(\mathrm{tr}\, w'w).$$

So, the joint distribution of $\ell_1,\ldots,\ell_{\bar{s}}$ follows in similar way as the joint distribution of $d_1,\ldots,d_s$.

6. ASYMPTOTIC DISTRIBUTIONS OF TEST STATISTICS WHEN THE OBSERVATIONS ARE INDEPENDENT

Consider the model

$$X = A\theta + E$$

where A and θ are as defined in (2.1), and

$$\underset{n\times p}{E} = \begin{pmatrix} \underset{\sim}{E}'_{(1)} \\ \vdots \\ \underset{\sim}{E}'_{(n)} \end{pmatrix}.$$

But we assume that $\underset{\sim}{E}_{(1)},\ldots,\underset{\sim}{E}_{(n)}$ are distributed independently as

$$|\Sigma|^{-1/2} h(\mathrm{tr}\Sigma^{-1}\underset{\sim}{E}_{(i)}\underset{\sim}{E}'_{(i)}) \tag{6.1}$$

with characteristic function $\phi(\mathrm{tr}\Sigma\underset{\sim}{T}\underset{\sim}{T}')$. We discuss the asymptotic distribution of the last s-r nonzero eigenvalues of $S_h(\hat{\theta},M)\{S_e(I,S)\}^{-1}$ and $S_f(\hat{\theta},M)\{S_e(I,S)\}^{-1}$.

Following the same lines as in Section 5, we have

$$\lim_{n\to\infty}|E - \tau \frac{1}{n} B_{\ell+1\,\ell+1}| = 0 \tag{6.2}$$

where

$$E = (\sum_{t=r+1}^{u} w_{1t\alpha}w_{1t\beta}) \quad \alpha,\beta = r+1,\ldots,p. \tag{6.3}$$

$$B_{\ell+1\ell+1} = (\sum_{t=1}^{n-k} w_{3t\alpha}w_{3t\beta}) \quad \alpha,\beta = r+1,\ldots,p. \tag{6.4}$$

$$\begin{aligned} \underset{u\times p}{W_1} &= (w_{1\alpha\beta}) = Y_{1*} - \Lambda \\ &= V_2[(I_n \;\; 0)\Gamma_C(I_k \;\; 0)\Gamma'_A(X-A\theta)Q']V'_1 \end{aligned}$$

$$\underset{(n-k)\times p}{W_3} = (w_{3\alpha\beta})$$

$$= V_2[(0 \; I_{n-k})\Gamma_A'(X-A\theta)Q']V_1' \tag{6.5}$$

and V_1, V_2 are two orthogonal matrices, $Q'Q = \Sigma^{-1} > 0$.

Let

$$\underset{n\times p}{Z} = \begin{pmatrix} \underset{\sim}{Z}'_{(1)} \\ \vdots \\ \underset{\sim}{Z}'_{(n)} \end{pmatrix} = (X-A\theta)Q' \tag{6.6}$$

From (6.1) we observe that $\underset{\sim}{Z}_{(1)},\ldots,\underset{\sim}{Z}_{(n)}$ are distributed independently as $h(\mathrm{tr}\underset{\sim\sim}{ZZ'})$.

Now, let

$$\underset{k\times k}{\Gamma_C} = (\Gamma_{Cij}), \quad \underset{n\times n}{\Gamma_A} = (\Gamma_{Ak\ell}).$$

By simple computations we have

$$W_1 = V_2 \begin{pmatrix} \sum_{\ell=1}^{n}(\sum_{i=1}^{k}\Gamma_{C1i}\Gamma_{A\ell i})Z_{\ell 1} & \cdots & \sum_{\ell=1}^{n}(\sum_{i=1}^{k}\Gamma_{C1i}\Gamma_{A\ell i})Z_{\ell n} \\ \vdots & & \vdots \\ \sum_{\ell=1}^{n}(\sum_{i=1}^{k}\Gamma_{Cui}\Gamma_{A\ell i})Z_{\ell 1} & \cdots & \sum_{\ell=1}^{n}(\sum_{i=1}^{k}\Gamma_{Cui}\Gamma_{A\ell i})Z_{\ell n} \end{pmatrix} V_1'$$

$$= V_2 \begin{bmatrix} Z^*_{11} & \cdots & Z^*_{1n} \\ \vdots & & \vdots \\ Z^*_{u1} & \cdots & Z^*_{un} \end{bmatrix} V_1' \quad \text{(say)} \tag{6.7}$$

and for any a (1,...,u), b (1,...,n) we have

$$E\,Z^*_{ab} = 0 \tag{6.8}$$

$$\text{Var } Z^*_{ab} = \text{Var}[\sum_{\ell=1}^{n}(\sum_{i=1}^{k}\Gamma_{Cai}\Gamma_{A\ell i})Z_{\ell b}] \tag{6.9}$$

$$= \sum_{\ell=1}^{n}(\sum_{i=1}^{k}\Gamma_{Cai}\Gamma_{A\ell i})^2(\text{var } Z_{\ell b})$$

$$= [-2\phi'(0)] \sum_{\ell=1}^{n}(\sum_{i=1}^{k}\Gamma^2_{Cai}\Gamma^2_{A\ell i} + \sum_{i\neq j}^{k}\Gamma_{Cai}\Gamma_{A\ell i}\Gamma_{Cbj}\Gamma_{A\ell j})$$

$$= [-2\phi'(0)]\{\sum_{i=1}^{k}\Gamma^2_{Cai}(\sum_{\ell=1}^{n}\Gamma^2_{A\ell i}) + \sum_{i\neq j}^{k}\Gamma_{Cai}\Gamma_{Cbj}(\sum_{\ell=1}^{n}\Gamma_{A\ell i}\Gamma_{A\ell j})\}$$

$$= -2\phi'(0). \tag{6.10}$$

In the following we prove that the different Z^*_{ab}'s are uncorrelated. For any $c(\neq a) \in (1,\ldots,u)$, $d(\neq b) \in (1,\ldots,n)$

$$E\, Z^*_{ab}Z^*_{cd} = E(\sum_{\ell=1}^{n}(\sum_{i=1}^{k}\Gamma_{Cai}\Gamma_{A\ell i})Z_{\ell b})(\sum_{\ell=1}^{n}(\sum_{i=1}^{k}\Gamma_{CCi}\Gamma_{A\ell i})Z_{\ell d})$$

$$= \sum_{\ell,m=1}[(\sum_{i=1}^{k}\Gamma_{Cai}\Gamma_{A\ell i})(\sum_{i=1}^{k}\Gamma_{CCi}\Gamma_{Ami})E(Z_{\ell b}Z_{md})]$$

When $\ell \neq m$ $E(Z_{\ell b}Z_{md}) = 0$ and so $E(Z^*_{ab}Z^*_{cd}) = 0$. When $\ell = m$, $b \neq d$ and $E(Z_{\ell b}Z_{md}) = 0$. So $E(Z^*_{ab}Z^*_{cd}) = 0$. When $\ell = m$, $b = d$, we have

$$E(Z^*_{ab}Z^*_{cd}) = (-2\phi'(0))\sum_{ij=1}^{k}\Gamma_{cai}\Gamma_{ccj}(\sum_{\ell=1}^{n}\Gamma_{A\ell i}\Gamma_{A\ell j}).$$

Also,

$$\sum_{\ell=1}^{n}\Gamma_{A\ell i}\Gamma_{A\ell j} = 0 \quad i \neq j$$

$$\sum_{\ell=1}^{n}\Gamma^2_{A\ell i} = 1, \text{ but } \sum_{i=1}^{k}\Gamma_{cai}\Gamma_{cci} = 0 \quad i = j.$$

So, $E(Z^*_{ab}Z^*_{cd}) = 0$. Now, let $Z^*_{ab} = \sqrt{-2\phi'(0)}\, U_{ab}$. By the central limit theorem we get that U_{ab}'s are mutually asymptotically independent normal variates with zero mean and unit standard deviation.

Similarly,

$$W_3 = V_2 \begin{pmatrix} \sum_{i=1}^{n} \Gamma_{Aik+1} Z_{i1} & \cdots & \sum_{i=1}^{n} \Gamma_{Ai,k+1} Z_{ip} \\ \cdots\cdot & \cdots & \cdots \\ \sum_{i=1}^{n} \Gamma_{Ain} Z_{i1} & \cdots & \sum_{i=1}^{n} \Gamma_{Ain} Z_{ip} \end{pmatrix} V_1'$$

$$= V_2 \begin{pmatrix} \tilde{Z}_{k+1,1} & \cdots & \tilde{Z}_{k+1,p} \\ \cdots\cdot & \cdots & \cdot \\ \tilde{Z}_{n1} & \cdots & \tilde{Z}_{np} \end{pmatrix} V_1' \quad \text{(say)}$$

and for any $i,k \in (k+1,\ldots,n)$ $j,\ell \in (1,\ldots,p)$

$$E\,\tilde{Z}_{ij} = 0$$

$$E\,\tilde{Z}_{ij}\tilde{Z}_{k\ell} = \begin{cases} -2\phi'(0) & i=k,\ j=\ell \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\tilde{Z}_{ij} = \sqrt{-2\phi'(0)}\, V_{ij}.$$

The V_{ij}'s are mutually asymptotically independent normal variates with zero mean and unit standard deviations.

Let

$$E^* = (\sum_{t=r+1}^{u} U_{t\alpha} U_{t\beta}) \quad \alpha,\beta = r+1,\ldots,p.$$

$$B^*_{\ell+1\ell+1} = (\sum_{t=1}^{n-k} V_{t\alpha} V_{t\beta}) \quad \alpha,\beta = r+1,\ldots,p.$$

Then

$$\lim_{n\to\infty} |E - \tau \frac{1}{n} B_{\ell+1\ell+1}| \quad 0$$

is equivalent to

$$\lim_{n\to\infty} \left| E^* - \tau \frac{1}{n} B^*_{\ell+1\ell+1} \right| = 0$$

and

$$\lim_{n\to\infty} \frac{1}{n} B^*_{\ell+1\ell+1} = I.$$

So, as $n \to \infty$, $\tau_{r+1},\ldots,\tau_s$ are the eigenvalues of the central Wishrat matrix with (u-r) degree of freedom.

Then following the same lines we can discuss the asymptotic distribution of the last s-r nonzero eigenvalues of $S_f(\hat{\theta},M)\{S_e(I,S)\}^{-1}$.

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In this paper, the authors derive likelihood ratio tests for the dimensionality of the regression matrices for the cases when the joint distributions of the observations are real and complex elliptically symmetric. The authors also derive asymptotic distributions of the above test statistics for two situations. In the first situation, the joint distribution of the observations is elliptically symmetric whereas the second situation assumes that the observations are distributed independently as elliptically symmetric.

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